

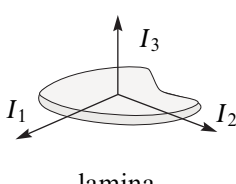
3.5 Rigid body dynamics

Moment of inertia tensor

Moment of inertia tensor ^a	$I_{ij} = \int (r^2 \delta_{ij} - x_i x_j) dm \quad (3.136)$	$r^2 = x^2 + y^2 + z^2$ δ_{ij} Kronecker delta $\mathbf{I}$ moment of inertia tensor dm mass element x_i position vector of dm I_{ij} components of $\mathbf{I}$
	$\mathbf{I} = \begin{pmatrix} \int (y^2 + z^2) dm & -\int xy dm & -\int xz dm \\ -\int xy dm & \int (x^2 + z^2) dm & -\int yz dm \\ -\int xz dm & -\int yz dm & \int (x^2 + y^2) dm \end{pmatrix} \quad (3.137)$	
Parallel axis theorem	$I_{12} = I_{12}^* - ma_1 a_2 \quad (3.138)$	I_{ij}^* tensor with respect to centre of mass
	$I_{11} = I_{11}^* + m(a_2^2 + a_3^2) \quad (3.139)$	$a_i, \mathbf{a}$ position vector of centre of mass
	$I_{ij} = I_{ij}^* + m(\mathbf{a} ^2 \delta_{ij} - a_i a_j) \quad (3.140)$	m mass of body
Angular momentum	$\mathbf{J} = \mathbf{I}\boldsymbol{\omega} \quad (3.141)$	$\mathbf{J}$ angular momentum $\boldsymbol{\omega}$ angular velocity
Rotational kinetic energy	$T = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{J} = \frac{1}{2} I_{ij} \omega_i \omega_j \quad (3.142)$	T kinetic energy

^a I_{ii} are the moments of inertia of the body. I_{ij} ($i \neq j$) are its products of inertia. The integrals are over the body volume.

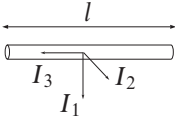
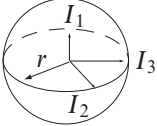
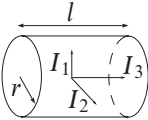
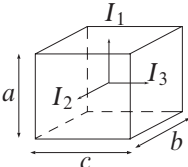
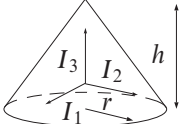
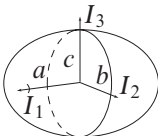
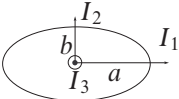
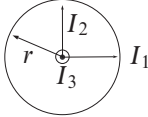
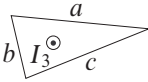
Principal axes

Principal moment of inertia tensor	$\mathbf{I}' = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \quad (3.143)$	$\mathbf{I}'$ principal moment of inertia tensor I_i principal moments of inertia
Angular momentum	$\mathbf{J} = (I_1 \omega_1, I_2 \omega_2, I_3 \omega_3) \quad (3.144)$	$\mathbf{J}$ angular momentum ω_i components of $\boldsymbol{\omega}$ along principal axes
Rotational kinetic energy	$T = \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2) \quad (3.145)$	T kinetic energy
Moment of inertia ellipsoid ^a	$T = T(\omega_1, \omega_2, \omega_3) \quad (3.146)$	
	$J_i = \frac{\partial T}{\partial \omega_i} \quad (\mathbf{J} \text{ is } \perp \text{ ellipsoid surface}) \quad (3.147)$	
Perpendicular axis theorem	$I_1 + I_2 \begin{cases} \geq I_3 & \text{generally} \\ = I_3 & \text{flat lamina } \perp \text{ to 3-axis} \end{cases} \quad (3.148)$	
Symmetries	$I_1 \neq I_2 \neq I_3 \quad \text{asymmetric top} \quad (3.149)$	
	$I_1 = I_2 \neq I_3 \quad \text{symmetric top}$	
	$I_1 = I_2 = I_3 \quad \text{spherical top}$	

^aThe ellipsoid is defined by the surface of constant T .



Moments of inertia^a

Thin rod, length l	$I_1 = I_2 = \frac{ml^2}{12}$ $I_3 \simeq 0$	(3.150) (3.151)	
Solid sphere, radius r	$I_1 = I_2 = I_3 = \frac{2}{5}mr^2$	(3.152)	
Spherical shell, radius r	$I_1 = I_2 = I_3 = \frac{2}{3}mr^2$	(3.153)	
Solid cylinder, radius r , length l	$I_1 = I_2 = \frac{m}{4} \left(r^2 + \frac{l^2}{3} \right)$ $I_3 = \frac{1}{2}mr^2$	(3.154) (3.155)	
Solid cuboid, sides a, b, c	$I_1 = m(b^2 + c^2)/12$ $I_2 = m(c^2 + a^2)/12$ $I_3 = m(a^2 + b^2)/12$	(3.156) (3.157) (3.158)	
Solid circular cone, base radius r , height h ^b	$I_1 = I_2 = \frac{3}{20}m \left(r^2 + \frac{h^2}{4} \right)$ $I_3 = \frac{3}{10}mr^2$	(3.159) (3.160)	
Solid ellipsoid, semi-axes a, b, c	$I_1 = m(b^2 + c^2)/5$ $I_2 = m(c^2 + a^2)/5$ $I_3 = m(a^2 + b^2)/5$	(3.161) (3.162) (3.163)	
Elliptical lamina, semi-axes a, b	$I_1 = mb^2/4$ $I_2 = ma^2/4$ $I_3 = m(a^2 + b^2)/4$	(3.164) (3.165) (3.166)	
Disk, radius r	$I_1 = I_2 = mr^2/4$ $I_3 = mr^2/2$	(3.167) (3.168)	
Triangular plate ^c	$I_3 = \frac{m}{36}(a^2 + b^2 + c^2)$	(3.169)	

^aWith respect to principal axes for bodies of mass m and uniform density. The radius of gyration is defined as $k = (I/m)^{1/2}$.

^bOrigin of axes is at the centre of mass ($h/4$ above the base).

^cAround an axis through the centre of mass and perpendicular to the plane of the plate.

Centres of mass

Solid hemisphere, radius r	$d = 3r/8$ from sphere centre	(3.170)
Hemispherical shell, radius r	$d = r/2$ from sphere centre	(3.171)
Sector of disk, radius r , angle 2θ	$d = \frac{2}{3}r \frac{\sin \theta}{\theta}$ from disk centre	(3.172)
Arc of circle, radius r , angle 2θ	$d = r \frac{\sin \theta}{\theta}$ from circle centre	(3.173)
Arbitrary triangular lamina, height h^a	$d = h/3$ perpendicular from base	(3.174)
Solid cone or pyramid, height h	$d = h/4$ perpendicular from base	(3.175)
Spherical cap, height h , sphere radius r	solid: $d = \frac{3}{4} \frac{(2r-h)^2}{3r-h}$ from sphere centre	(3.176)
	shell: $d = r - h/2$ from sphere centre	(3.177)
Semi-elliptical lamina, height h	$d = \frac{4h}{3\pi}$ from base	(3.178)

^a h is the perpendicular distance between the base and apex of the triangle.

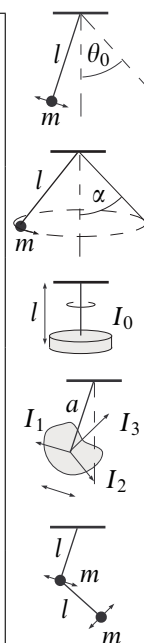
Pendulums

Simple pendulum	$P = 2\pi \sqrt{\frac{l}{g} \left(1 + \frac{\theta_0^2}{16} + \dots \right)}$	(3.179)	P period g gravitational acceleration l length θ_0 maximum angular displacement
Conical pendulum	$P = 2\pi \left(\frac{l \cos \alpha}{g} \right)^{1/2}$	(3.180)	α cone half-angle
Torsional pendulum ^a	$P = 2\pi \left(\frac{I_0}{C} \right)^{1/2}$	(3.181)	I_0 moment of inertia of bob C torsional rigidity of wire (see page 81)
Compound pendulum ^b	$P \simeq 2\pi \left[\frac{1}{mga} (ma^2 + I_1 \cos^2 \gamma_1 + I_2 \cos^2 \gamma_2 + I_3 \cos^2 \gamma_3) \right]^{1/2}$	(3.182)	a distance of rotation axis from centre of mass m mass of body I_i principal moments of inertia γ_i angles between rotation axis and principal axes
Equal double pendulum ^c	$P \simeq 2\pi \left[\frac{l}{(2 \pm \sqrt{2})g} \right]^{1/2}$	(3.183)	

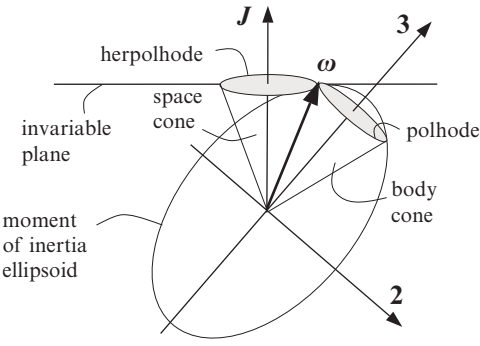
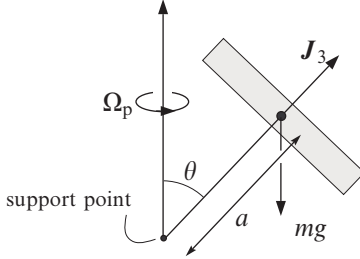
^aAssuming the bob is supported parallel to a principal rotation axis.

^bI.e., an arbitrary triaxial rigid body.

^cFor very small oscillations (two eigenmodes).



Tops and gyroscopes

 prolate symmetric top			 gyroscope		
Euler's equations ^a	$G_1 = I_1 \dot{\omega}_1 + (I_3 - I_2) \omega_2 \omega_3 \quad (3.184)$ $G_2 = I_2 \dot{\omega}_2 + (I_1 - I_3) \omega_3 \omega_1 \quad (3.185)$ $G_3 = I_3 \dot{\omega}_3 + (I_2 - I_1) \omega_1 \omega_2 \quad (3.186)$		G_i	external couple (=0 for free rotation)	
			I_i	principal moments of inertia	
			ω_i	angular velocity of rotation	
Free symmetric top ^b ($I_3 < I_2 = I_1$)	$\Omega_b = \frac{I_1 - I_3}{I_1} \omega_3 \quad (3.187)$ $\Omega_s = \frac{J}{I_1} \quad (3.188)$		Ω_b	body frequency	
			Ω_s	space frequency	
			J	total angular momentum	
Free asymmetric top ^c	$\Omega_b^2 = \frac{(I_1 - I_3)(I_2 - I_3)}{I_1 I_2} \omega_3^2 \quad (3.189)$				
Steady gyroscopic precession	$\Omega_p^2 I_1' \cos \theta - \Omega_p J_3 + m g a = 0 \quad (3.190)$ $\Omega_p \simeq \begin{cases} M g a / J_3 & (\text{slow}) \\ J_3 / (I_1' \cos \theta) & (\text{fast}) \end{cases} \quad (3.191)$		Ω_p	precession angular velocity	
			θ	angle from vertical	
			J_3	angular momentum around symmetry axis	
			m	mass	
			g	gravitational acceleration	
			a	distance of centre of mass from support point	
			I_1'	moment of inertia about support point	
Gyroscopic stability	$J_3^2 \geq 4 I_1' m g a \cos \theta \quad (3.192)$				
Gyroscopic limit ("sleeping top")	$J_3^2 \gg I_1' m g a \quad (3.193)$				
Nutation rate	$\Omega_n = J_3 / I_1' \quad (3.194)$		Ω_n	nutation angular velocity	
Gyroscope released from rest	$\Omega_p = \frac{m g a}{J_3} (1 - \cos \Omega_n t) \quad (3.195)$		t	time	

^aComponents are with respect to the principal axes, rotating with the body.

^bThe body frequency is the angular velocity (with respect to principal axes) of ω around the 3-axis. The space frequency is the angular velocity of the 3-axis around J , i.e., the angular velocity at which the body cone moves around the space cone.

^c J close to 3-axis. If $\Omega_b^2 < 0$, the body tumbles.